

Poisson homogeneous spaces

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Abstract

General framework for Poisson homogeneous spaces of Poisson groups is introduced. Poisson Minkowski spaces are discussed as a particular example.

Introduction

Poisson Lie groups, introduced in [1] (see also [2, 3, 4, 5]), allow one to describe generalized symmetries of classical mechanical systems and provide an important tool for studying quantum deformations of Lie groups. The classification of Poisson structures on a given group gives usually a correct idea about the classification of the quantum deformations of this group.

Together with Poisson (or quantum) groups, one has to consider Poisson (or quantum) spaces on which these groups act. This, in particular, rises the question of the classification of homogeneous Poisson (or quantum) spaces for a given Poisson (quantum) group.

In this paper we collect some fundamental facts concerning Poisson homogeneous spaces and apply this to the classification of Poisson Minkowski spaces. For each Poisson structure on the Poincaré group (as studied recently in [6]) there is exactly one Poisson structure on the Minkowski space, making it a Poisson homogeneous space.

1 Preliminaries

We collect here basic definitions related to Poisson manifolds [7] and Poisson groups (main references: [1, 2, 3, 4, 5]). We start with introducing a simple notation.

Notation.

1. Let M, N be manifolds and $f: M \rightarrow N$ a smooth map. For any contravariant tensor ξ at some point of M we denote by $f(\xi)$ the image of ξ using f . More precisely, if $\xi \in \otimes^k T_x^* M$, then $f(\xi)$ will denote $(\otimes^k T_x f)(\xi)$.

2. Let M, N, P be manifolds and $f: M \times N \rightarrow P$ a smooth map. Let $_x f: N \rightarrow P$, $f_y: M \rightarrow P$ be defined by

$$_x f(x) = f(x, y) = f_y(x) \quad \text{for } x \in M, y \in N$$

and suppose we denote f as a product, $f(x, y) = xy$. In this case we shall use the following notation:

$$\xi y := f_y(\xi), \quad x\eta := {}_x f(\eta),$$

if ξ is a contravariant tensor at some point of M , η – a contravariant tensor at some point of N .

A bivector field π on a (smooth, real) manifold M is said to be *Poisson*, if its associated bracket of functions,

$$\{f, g\}_\pi = \langle df \wedge dg, \pi \rangle \quad (1)$$

satisfies the Jacobi identity (in this case the bracket is said to be *Poisson*). A bivector field π is Poisson if and only if the Schouten bracket $[\pi, \pi]$ is zero. Poisson bivectors are also called *Poisson structures*. A *Poisson manifold* is a pair (M, π) , where M is a manifold and π is a Poisson structure on M .

Let (M, π) , (N, ρ) be two Poisson manifolds. A smooth map $f: M \rightarrow N$ is said to be a *Poisson map* if $f(\pi(x)) = \rho(f(x))$ for each $x \in M$.

If (M, π) is a Poisson manifold and $f: M \rightarrow N$ a surjective submersion such that $f(\pi(x)) = f(\pi(x'))$ whenever $f(x) = f(x')$, then there is exactly one Poisson structure ρ on N such that f is a Poisson map. It is given by $\rho(f(x)) := f(\pi(x))$.

Poisson product $(M, \pi) \times (N, \rho)$ of two Poisson manifolds (M, π) and (N, ρ) is the Poisson manifold $(M \times N, \sigma)$ with the Poisson structure σ given by

$$\sigma(x, y) := \pi(x) \oplus \rho(y) \in \bigwedge^2 T_x M \oplus \bigwedge^2 T_y N \subset \bigwedge^2 T_{(x,y)}(M \times N) \quad \text{for } (x, y) \in M \times N.$$

A *Poisson group* is a Poisson manifold (G, π) together with a Poisson map $m: G \times G \rightarrow G$ (the product Poisson structure on $G \times G$), such that (G, m) is a group (consequently, this group is a Lie group).

Since $m(X \oplus Y) = Xh + gY$ for $X \in T_g G$, $Y \in T_h G$, the map m is Poisson if and only if it is *multiplicative*:

$$\pi(gh) = \pi(g)h + g\pi(h) \quad \text{for } g, h \in G, \quad (2)$$

where $gh := m(g, h)$.

Let (G, π) be a Poisson group. We denote by \mathfrak{g} the Lie algebra of G . Since $\pi(e) = 0$, where e is the group unit, the bivector field π has a well defined linearization at e . The linearization of π at e will be denoted by δ . It is a linear map from \mathfrak{g} to $\bigwedge^2 \mathfrak{g}$, called *cocommutator* or *cobacket*. The linear map $\delta: \mathfrak{g} \rightarrow \bigwedge^2 \mathfrak{g}$ satisfies two conditions:

1. it is a 1-cocycle on \mathfrak{g} with values in $\bigwedge^2 \mathfrak{g}$ (with respect to the adjoint action)
2. it is a (linear) Poisson bivector on \mathfrak{g}

Any pair (\mathfrak{g}, δ) , where $\delta: \mathfrak{g} \rightarrow \bigwedge^2 \mathfrak{g}$ is a linear map satisfying two above conditions, is said to be a *Lie bialgebra*. The above mentioned correspondence between Poisson groups and Lie bialgebras is known to be one to one, if we consider only Poisson groups (G, π) for which G is connected and simply connected.

2 Poisson homogeneous spaces

If (G, π) is a Poisson group and (M, π_M) is a Poisson manifold then an action $\phi: G \times M \rightarrow M$ is said to be a *Poisson action* if ϕ is a Poisson map. It is true if and only if π_M is (G, π) -multiplicative in the following sense:

$$\pi_M(gx) = \pi(g)x + g\pi_M(x) \quad \text{for } g \in G, x \in M, \quad (3)$$

where $gx := \phi(g, x)$. If the action ϕ is transitive then (M, π) is said to be a *Poisson homogeneous space* of (G, π) .

In this section we prepare some tools to construct Poisson homogeneous spaces of a given Poisson group. In particular, we shall be interested in the following problem.

Problem 2.1 *Given a Poisson group (G, π) and a transitive action of G on M , find all Poisson structures on M such that the action is Poisson.*

We start with two simple observations:

1. The difference of two (G, π) -multiplicative bivector fields on M is G -invariant. The sum of a (G, π) -multiplicative bivector field and of a G -invariant bivector field is (G, π) -multiplicative. (The space of (G, π) -multiplicative bivector fields is an affine space modelled on the space of G -invariant bivector fields.)
2. A (G, π) -multiplicative bivector field π_M on M which vanishes at some point is automatically Poisson.

The second observation follows from the fact that if a (G, π) -multiplicative π_M vanishes at $x_0 \in M$, then

$$\pi_M(gx_0) = \pi(g)x_0$$

hence π_M is the image of π under the orbital map $g \mapsto gx_0$ (a surjective submersion).

In order to illustrate these facts, we consider first a special case. The Poisson group (G, π) is fixed throughout this section.

2.1 Affine homogeneous space

We consider G acting on itself by left translations. We are interested in (G, π) -multiplicative Poisson bivector fields α on G . Since π is a solution of this problem, an arbitrary solution α is a Poisson bivector field which differs from π by a left-invariant field $g \mapsto A^l(g) := gA$, $A \in \bigwedge^2 \mathfrak{g}$, hence

$$\alpha = \pi + A^l, \quad (4)$$

for some $A \in \bigwedge^2 \mathfrak{g}$ (in fact, $A = \alpha(e)$). Any α of the form (4) is (G, π) -multiplicative. It is Poisson if and only if

$$0 = [\alpha, \alpha] = [\pi + A^l, \pi + A^l] = 2[\pi, A^l] + [A^l, A^l]. \quad (5)$$

It is easy to see that the right-hand side is always left-invariant ($g\pi = \pi + B^r$, where B^r is some right-invariant field), hence it is zero if and only if

$$2\delta(A) + [A, A] = 0, \quad (6)$$

where $\delta(A) := [\pi, A^l](e)$, $[A, A] := [A^l, A^l](e)$ (since $\pi(e) = 0$, $[\pi, A^l](e)$ depends only on the value of A^l at e and on the linearization δ of π).

We conclude that (G, π) -multiplicative Poisson bivector fields α on G are in 1–1 correspondence with solutions A of (6).

Remark 2.2 Poisson structures α on G such that

$$\alpha - (\alpha(e))^l = \pi$$

are exactly *affine Poisson structures* on G in the sense of [8, 5], whose *associated left multiplicative Poisson structure* is π . Condition (6) has been derived in [8, 5]. It is also shown in [5], that the solutions of (6) are in 1–1 correspondence with isotropic Lie subalgebras which are complementary to \mathfrak{g} in the Manin triple Lie algebra $\mathfrak{g} \ltimes \mathfrak{g}^*$.

Special case. It is easy to describe all (G, π) -multiplicative Poisson bivector fields α on G which vanish at some point. Since $\alpha(g) = \pi(g) + g\alpha(e)$, assuming $\alpha(g_0) = 0$ we obtain $\alpha(e) = -g_0^{-1}\pi(g_0)$ and

$$\alpha(g) = \pi(g) - gg_0^{-1}\pi(g_0) = \pi(gg_0^{-1}g_0) - gg_0^{-1}\pi(g_0) = \pi(gg_0^{-1})g_0 = (\pi g_0)(g),$$

hence α is just the right translation of π by g_0 . (The corresponding isotropic complementary Lie subalgebra in the Manin triple is just the image of \mathfrak{g}^* by the adjoint action of g_0 in $\mathfrak{g} \ltimes \mathfrak{g}^*$.)

2.2 General situation

We return to the situation of a general transitive action $(g, x) \mapsto gx$ of G on a manifold M .

First we formulate a result which allows to localize and linearize the problem. The subgroup in G stabilizing an element $x_0 \in M$ is denoted by G_{x_0} . The Lie algebra of G_{x_0} is denoted by \mathfrak{g}_{x_0} .

Lemma 2.3 *For any $x_0 \in M$ there is a 1–1 correspondence between*

1. *G -multiplicative bivector fields π_M on M*

and

2. *elements $\rho \in \bigwedge^2 T_{x_0}M$ such that*

$$\rho = \pi(h)x_0 + h\rho \quad \text{for } h \in G_{x_0}. \quad (7)$$

The correspondence is given by:

$$\rho := \pi_M(x_0) \quad (8)$$

$$\pi_M(x) := \pi(g)x_0 + g\rho \quad \text{where } g \text{ is any element of } G \text{ such that } x = gx_0. \quad (9)$$

Proof: If π_M is (G, π) -multiplicative then $\rho = \pi_M(x_0)$ satisfies (7). Conversely, if ρ satisfies (7) then π_M is well defined by (9), since

$$\pi(gh)x_0 + gh\rho = \pi(g)hx_0 + g\pi(h)x_0 + g(\rho - \pi(h)x_0) = \pi(g)x_0 + g\rho$$

for $g \in G$, $h \in G_{x_0}$. The (G, π) -multiplicativity follows from

$$\pi_M(gx) = \pi_M(gg_x x_0) = \pi(gg_x)x_0 + gg_x\rho = \pi(g)g_x x_0 + g(\pi(g_x)x_0 + g_x\rho) = \pi(g)x + g\pi_M(x),$$

where $g_x \in G$ is such that $x = g_x x_0$.

Q.E.D.

Corollary. For connected G_{x_0} we have a 1–1 correspondence between

1. (G, π) -multiplicative bivector fields π_M on M and
2. elements $\rho \in \bigwedge^2 T_{x_0} M$ such that

$$(\delta(X))x_0 + X\rho = 0 \quad \text{for } X \in \mathfrak{g}_{x_0}. \quad (10)$$

Denoting by $\tilde{\rho}$ any lift of ρ to $\bigwedge^2 \mathfrak{g}$,

$$\tilde{\rho}x_0 = \rho,$$

we can write condition (10) in the following equivalent form:

$$\delta(X) + \text{ad}_X \tilde{\rho} \in \mathfrak{g} \wedge \mathfrak{g}_{x_0} \quad \text{for } X \in \mathfrak{g}_{x_0}. \quad (11)$$

Now suppose ρ satisfies (7) hence the corresponding π_M defined by (9) is (G, π) -multiplicative. Let $\tilde{\rho} \in \bigwedge^2 \mathfrak{g}$ be such that $\tilde{\rho}x_0 = \rho$. Since

$$\pi_M(gx_0) = (\pi(g) + g\tilde{\rho})x_0,$$

π_M is the image of $\pi + \tilde{\rho}^\flat$ by the canonical projection (the orbital map) from G to $G/G_{x_0} = M$. It is clear that

$$[\pi_M, \pi_M] = 0 \quad \Longleftrightarrow \quad [\pi + \tilde{\rho}^\flat, \pi + \tilde{\rho}^\flat]x_0 = 0.$$

Since $[\pi + \tilde{\rho}^\flat, \pi + \tilde{\rho}^\flat]$ is left-invariant (as in formula (5)), the last equality is equivalent to

$$(2\delta(\tilde{\rho}) + [\tilde{\rho}, \tilde{\rho}])x_0 = 0, \quad (12)$$

or,

$$2\delta(\tilde{\rho}) + [\tilde{\rho}, \tilde{\rho}] \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}_{x_0}. \quad (13)$$

Conclusion: solving problem 2.1 is equivalent to find ρ satisfying (7) (or (11) if G_{x_0} is connected) and (12). ($\tilde{\rho}$ is any lift of ρ to $\bigwedge^2 \mathfrak{g}$.)

2.3 The coboundary case

In this subsection we suppose that δ is a coboundary of some (classical r -matrix) $r \in \overset{2}{\bigwedge} \mathfrak{g}$:

$$\delta(X) = -\text{ad}_X r.$$

In this case, for $A \in \overset{2}{\bigwedge} \mathfrak{g}$ we have

$$\delta(A) = [\pi, A^l](e) = [r^{\text{right}} - r^l, A^l](e) = -[r, A],$$

where r^{right} is the right-invariant field on G corresponding to r . Conditions (11) and (13) take now the form

$$\text{ad}_X(\tilde{\rho} - r) \in \mathfrak{g} \wedge \mathfrak{g}_{x_0} \quad \text{for } X \in \mathfrak{g}_{x_0} \quad (14)$$

and

$$([\tilde{\rho}, \tilde{\rho}] - 2[r, \tilde{\rho}]) \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}_{x_0}, \quad (15)$$

respectively.

2.3.1 The case of the Poisson quotient

Condition (11) has as solution $\tilde{\rho} = 0$ if and only if

$$\delta(\mathfrak{g}_{x_0}) \subset \mathfrak{g} \wedge \mathfrak{g}_{x_0}. \quad (16)$$

This is exactly the case when π is projectable by the canonical projection from G to $G/G_{x_0} = M$. The image of π by this projection is then the (G, π) -multiplicative π_M defined by $\pi_M(x_0) = \rho = 0$.

The subgroup G_{x_0} such that (16) is satisfied is called *coisotropic* in [5]. Another way of expressing condition (16) is to require the annihilator $\mathfrak{g}_{x_0}^\perp$ of \mathfrak{g}_{x_0} to be a Lie subalgebra in \mathfrak{g}^* (with respect to the bracket defined by the dual map of δ).

A special case of this situation arises when G_{x_0} is a *Poisson subgroup* of G , i.e.

$$\delta(\mathfrak{g}_{x_0}) \subset \mathfrak{g}_{x_0} \wedge \mathfrak{g}_{x_0}$$

(equivalently: $\mathfrak{g}_{x_0}^\perp$ is an ideal in \mathfrak{g}^*), cf. [3, 4].

Example. Recall that the ‘standard’ Poisson structure π on a simple Lie group G is defined in such a way that the cobracket δ vanishes on some Cartan subalgebra $\mathfrak{h} \in \mathfrak{g}$ (the corresponding Cartan subgroup H is a ‘classical’ subgroup of the Poisson group G). It follows that the quotient G/H carries a natural Poisson structure (the reduction of π by the canonical projection) endowing G/H with the structure of a Poisson (G, π) -homogeneous space. In particular, taking $G = SU(2)$, $H = S^1 \subset SU(2)$ we obtain the ‘standard’ Poisson sphere. Any Poisson structure on the sphere which makes it a Poisson homogeneous $SU(2)$ -space is a sum of the standard Poisson structure and any $SU(2)$ invariant bivector field on the sphere (in two dimensions all bivector fields are Poisson), cf. [9].

More general cases were described in [4, 10].

3 Poisson Minkowski spaces

Poisson Poincaré groups in dimension $D = 4$ have been discussed and almost completely classified recently in [6]. They are all coboundary. The Lie algebra \mathfrak{g} has the semidirect structure

$$\mathfrak{g} = V \rtimes \mathfrak{h},$$

where $\mathfrak{h} \cong sl(2, \mathbb{C})$ is the Lorentz algebra and V is the subgroup of translations (which may be identified with the Minkowski space). The r -matrix has a decomposition

$$r = a \oplus b \oplus c \in \bigwedge^2 V \oplus (V \wedge \mathfrak{h}) \oplus \bigwedge^2 \mathfrak{h}.$$

We have a simple proposition.

Proposition 3.1 *For any classical r -matrix r on the Poincaré Lie algebra there is exactly one Poisson structure on $M := G/H$ such that the action of G on M is Poisson. This structure is determined by the condition $\rho \equiv \pi_M(x_0) = a$.*

Proof: The uniqueness follows from the fact that zero is the only G -invariant bivector field on M (there are no non-zero \mathfrak{h} -invariants in $\bigwedge^2 V$). We can always take the lift $\tilde{\rho}$ belonging to $\bigwedge^2 V$. Condition (14) reads

$$\text{ad}_X(\tilde{\rho} - a - b - c) \in \mathfrak{g} \wedge \mathfrak{h} \quad \text{for } X \in \mathfrak{h},$$

hence

$$\text{ad}_X(\tilde{\rho} - a) \in \mathfrak{g} \wedge \mathfrak{h} \quad \text{for } X \in \mathfrak{h},$$

and therefore $\tilde{\rho} = a$. Since $[c, a] \in (\bigwedge^2 V) \wedge \mathfrak{h}$ and $[b, a] = 0$ (cf. [6]), we have

$$[a, a] - 2[r, a] \in (\bigwedge^2 V) \wedge \mathfrak{h}$$

and condition (15) is satisfied. □

Remark 3.2 *The proposition has a straightforward generalization for the $\mathbb{R}^{p+q} \rtimes O(p, q)$ groups discussed in [6]. Next few remarks are also valid in more general context.*

Now we derive a practical formula to compute the Poisson structure on M corresponding to a given classical r -matrix on \mathfrak{g} . Due to the group law,

$$(v, \Lambda)(v', \Lambda') = (v + \Lambda v', \Lambda \Lambda'), \quad v, v' \in V, \Lambda, \Lambda' \in H,$$

the right and left translations of a vector $(u, X) \in \mathfrak{g} = V \rtimes \mathfrak{h}$ by $g = (v, I) \equiv v \in V$ are given by

$$(u, X)v = (u + Xv, X), \quad v(u, X) = (u, X). \quad (17)$$

Using formula (9) with $x_0 = 0$, $x = v = g$ and $\pi(g) = rg - gr$, we obtain

$$\pi(g)x_0 = \pi(v)x_0 = ((a + b + c)v - v(a + b + c))x_0 = (bv - vb)x_0 + (cv - vc)x_0,$$

hence

$$\pi_M(v) = (bv - vb)x_0 + (cv - vc)x_0 + a.$$

Writing b and c in the form $b = w_j \wedge Z_j$, $c = X_k \wedge Y_k$ (summation convention) with $w_j \in V$, $X_k, Y_k, Z_j \in \mathfrak{h}$, we obtain (using (17))

$$\begin{aligned}\pi_M(v) &= a + (w_j \wedge (Z_j v + Z_j) - w_j \wedge Z_j)x_0 + ((X_k v + X_k) \wedge (Y_k v + Y_k))x_0 \\ \pi_M(v) &= a + w_j \wedge Z_j v + X_k v \wedge Y_k v = a + b(v) + c(v \otimes v),\end{aligned}\tag{18}$$

with obvious notation. Therefore, a , b and c determine the constant, linear and quadratic part of π_M , respectively.

It is interesting to note, that a , b and c (hence r) can be recovered from π_M . Obviously, $a = \pi_M(0)$. In order to see that b and c are determined by π_M , let us write $b(v)$ and $c(v \otimes v)$ in more detail. Using the isomorphism

$$\bigwedge^2 V \cong \mathfrak{h}, \quad x \wedge y \mapsto \Omega_{x,y} := x \otimes \eta(y) - y \otimes \eta(x),$$

where $\eta: V \rightarrow V^*$ is the metric tensor, we can think of b as an element of $V \otimes \bigwedge^2 V \subset V \otimes V \otimes V$,

$$b = b^{\alpha\beta\gamma} e_\alpha \otimes e_\beta \otimes e_\gamma, \quad b^{\alpha\beta\gamma} = -b^{\alpha\gamma\beta}.$$

Here e_α is a basis of V . We have

$$b(v) = b^{\alpha\beta\gamma} e_\alpha \wedge (e_\beta \eta_{\gamma\mu} - e_\gamma \eta_{\beta\mu}) v^\mu = f^{\alpha\beta\gamma} e_\alpha \wedge e_\beta \eta_{\gamma\mu} v^\mu,$$

where

$$f^{\alpha\beta\gamma} := b^{\alpha\beta\gamma} - b^{\beta\alpha\gamma}.$$

One can easily see that b is computed from f (hence from π_m) as follows

$$b^{\alpha\beta\gamma} = \frac{1}{2}(f^{\alpha\beta\gamma} + f^{\gamma\alpha\beta} - f^{\beta\gamma\alpha}).$$

Similarly, we can think of $c \in \mathfrak{h} \wedge \mathfrak{h}$ as an element of $\bigwedge^2 V \wedge \bigwedge^2 V \subset V \otimes V \otimes V \otimes V$,

$$c = c^{\alpha\beta\gamma\delta} e_\alpha \otimes e_\beta \otimes e_\gamma \otimes e_\delta, \quad c^{\alpha\beta\gamma\delta} = -c^{\beta\alpha\gamma\delta} = -c^{\alpha\beta\delta\gamma} = -c^{\gamma\delta\alpha\beta}.$$

For the symmetric bilinear form corresponding to $c(v \otimes v)$ we get the following expression:

$$\frac{1}{2}(c(v \otimes u) + c(u \otimes v)) = h^{\alpha\beta\gamma\delta} e_\alpha \wedge e_\gamma \eta_{\beta\mu} \eta_{\delta\nu} v^\mu u^\nu,$$

where

$$h^{\alpha\beta\gamma\delta} = 2(c^{\alpha\beta\gamma\delta} + c^{\alpha\delta\gamma\beta}).$$

It turns out that c can be computed from h (hence from π_M):

$$c^{\alpha\beta\gamma\delta} = \frac{1}{4}(h^{\alpha\beta\gamma\delta} - h^{\delta\gamma\beta\alpha}).$$

Below we list the Poisson structures on M corresponding to the first six cases in the table of r -matrices for the Poincaré group given in [6] (the case of non-vanishing c). Here x^0, x^1, x^2, x^3 are the Lorentzian coordinates and $x^\pm := x^0 \pm x^3$. Not displayed brackets are zero.

$$1. \quad r = \gamma JH \wedge H + \alpha e_+ \wedge e_- + \tilde{\alpha} e_1 \wedge e_2$$

$$\begin{aligned} \{x^1, x^\pm\} &= \gamma x^2 x^\pm \\ \{x^2, x^\pm\} &= -\gamma x^1 x^\pm \\ \{x^+, x^-\} &= \alpha \\ \{x^1, x^2\} &= \tilde{\alpha} \end{aligned}$$

Note that the two first equalities can be written in the form

$$\{z, x^\pm\} = -i\gamma z x^\pm,$$

where $z := x^1 + ix^2$.

$$2. \quad r = JX_+ \wedge X_+ + \beta_1(e_1 \wedge X_+ - e_2 \wedge JX_+ + e_+ \wedge H) + \beta_2 e_+ \wedge JH$$

$$\begin{aligned} \{x^1, x^+\} &= 2x^2 x^- + \beta_1 x^1 - \beta_2 x^2 \\ \{x^2, x^+\} &= -2x^1 x^- + \beta_1 x^2 + \beta_2 x^1 \\ \{x^+, x^-\} &= -\beta_1 x^- \\ \{x^1, x^2\} &= 4(x^-)^2 \end{aligned}$$

$$3. \quad r = JX_+ \wedge X_+ + \beta(e_1 \wedge X_+ - e_2 \wedge JX_+ + e_+ \wedge H) + \alpha e_+ \wedge e_1$$

$$\begin{aligned} \{x^1, x^+\} &= 2x^2 x^- + \beta x^1 - \alpha \\ \{x^2, x^+\} &= -2x^1 x^- + \beta x^2 \\ \{x^+, x^-\} &= -\beta x^- \\ \{x^1, x^2\} &= 4(x^-)^2 \end{aligned}$$

$$4. \quad r = JX_+ \wedge X_+ + \beta(e_1 \wedge X_+ + e_2 \wedge JX_+ + e_+ \wedge (\alpha_1 e_1 + \alpha_2 e_2)) - \beta^2 e_1 \wedge e_2$$

$$\begin{aligned} \{x^1, x^+\} &= 2x^2 x^- + \beta x^1 - \alpha_1 \\ \{x^2, x^+\} &= -2x^1 x^- - \beta x^2 - \alpha_2 \\ \{x^1, x^2\} &= 4(x^-)^2 - \beta^2 \end{aligned}$$

$$5. \quad r = H \wedge X_+ - JH \wedge JX_+ + \gamma JX_+ \wedge X_+$$

$$\begin{aligned} \{x^1, x^+\} &= -2x^+ x^- - (x^2)^2 + 2\gamma x^2 x^- \\ \{x^2, x^+\} &= x^1 x^2 - 2\gamma x^1 x^- \\ \{x^1, x^-\} &= 2(x^-)^2 \\ \{x^1, x^2\} &= -2x^2 x^- + 4\gamma (x^-)^2 \\ \{x^+, x^-\} &= x^- x^1 \end{aligned}$$

$$6. \quad r = H \wedge X_+ + \beta e_2 \wedge X_+$$

$$\begin{aligned} \{x^1, x^+\} &= -2x^+ x^- \\ \{x^2, x^+\} &= \beta x^1 \\ \{x^1, x^-\} &= 2(x^-)^2 \\ \{x^1, x^2\} &= -2\beta x^- \\ \{x^+, x^-\} &= x^- x^1 \end{aligned}$$

Remark 3.3 $D = 2$ Poincaré groups and corresponding Poisson Minkowski spaces have been classified in [11].

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